January 2018, Q1

R = C[x, y, z]
I = (x, y)

a.) Prove that I is a prime ideal.

$$R/I \cong \mathbb{C}[z] \quad (\text{domain})$$

b.) Let J = (x^2, y^2). Prove that if f_1 … f_n belongs to J, then there is a subset of {f_1, …, f_n} of at most three polynomials whose product belongs to J.

$f_1 \cdots f_n \in J \Rightarrow f_1 \cdots f_n \in I \Longrightarrow f_1 \in I$ (after rearranging, if necessary) because I is prime

If f_1 belongs to J, then we are done. Suppose that this is not the case. In particular, there exist some polynomials a, b in R such that f_1 = ax + by and either a is not divisible by x or b is not divisible by y. Because f_1 … f_n belongs to J, there exist polynomials c, d in R such that ax f_2 … f_n + by f_2 … f_n = f_1 … f_n = cx^2 + dy^2.

$$x(af_2 \cdots f_n - cx) = y(dy - bf_2 \cdots f_n)$$

$x \mid (dy - bf_2 \cdots f_n) \Rightarrow bf_2 \cdots f_n \in I$ and $y \mid (af_2 \cdots f_n - cx) \Rightarrow af_2 \cdots f_n \in I$

$a, b \in I \Rightarrow \left.\begin{array}{l} a = ex + gy \\ b = hx + iy \end{array}\right\} \to f_1 = ex^2 + (g+h)xy + iy^2$

$f_1^2 \in J$ ✓

Proceed by cases. This method is tedious.
We can also use the multidegree method.

multidegrees: $\left(\deg_x a + \sum_{i=2}^{n} \deg_x f_i + 1, \ \deg_y b + \sum_{i=2}^{n} \deg_y f_i + 1\right)$

$$= (2 + \deg_x c, \ 2 + \deg_y d)$$

1.) If deg_x(a) = 1, then deg_y(f_2) (after rearranging). Otherwise, deg_x(f_2) = 1.

2.) If deg_y(b) = 1, then deg_x(f_3) = 1 (after rearranging). Otherwise, deg_y(f_3) = 1.

At any rate, the product f_1 f_2 f_3 belongs to J.

c.) Let K = (x^2 y^2, x^2 z^2, y^2 z^2). Prove that if f_1 … f_n belongs to K, then there is a subset of {f_1, …, f_n} of at most nine polynomials whose product belongs to K.

$I_{x,z} = (x, z)$, $J_{x,z} = (x^2, z^2)$, $I_{y,z} = (y, z)$, $J_{y,z} = (y^2, z^2)$

$$K = J_{x,y} \cdot J_{x,z} \cdot J_{y,z}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$f_1 f_2 f_3 \quad f_4 f_5 f_6 \quad f_7 f_8 f_9 \quad \text{by (b.)}$$

August 2016, Q1

Let R be a commutative unital ring with units U(R).

a.) Prove that U(R) is a multiplicative abelian group.

b.) Let R = Z[x]/(x^2). Prove that U(R) is isomorphic to Z x (Z/2Z).

$$R = \{ a\bar{x} + b\bar{1} \mid a, b \in \mathbb{Z} \}$$

$$(a\bar{x} + b\bar{1})(c\bar{x} + d\bar{1}) = (ad + bc)\bar{x} + bd\bar{1} \in U(R) \iff \begin{array}{l} ad+bc = 0 \\ bd = 1 \end{array} \iff \begin{array}{l} b = d = \pm 1 \\ a = \pm c \end{array}$$

$$U(R) = \{a\bar{x} \pm \bar{1} \mid a \in \mathbb{Z}\}$$

$$\varphi: U(R) \to \mathbb{Z} \times \mathbb{Z}_2$$

Because this is a group homomorphism, we need \varphi($\bar{1}$) = (0, 0) and \varphi(-$\bar{1}$) = (0, 1).

$$-\bar{1} = a\bar{x} - a\bar{x} - \bar{1} = (a\bar{x}+\bar{1})(a\bar{x}-\bar{1})$$

$$(0,1) = \varphi(-\bar{1}) = \varphi(a\bar{x}+\bar{1}) + \varphi(a\bar{x}-\bar{1})$$

$$= \underbrace{(a,0)} + \underbrace{(-a,1)}$$

$$\mathbb{Z}\times\{0\} \quad \cup \quad \mathbb{Z}\times\{1\}$$

Define \varphi(ax + 1) = (a, 0) and \varphi(ax - 1) = (-a, 1). This is an isomorphism.

January 2018, Q2

Let G = Z x Z.

a.) Give a nontrivial element (a, b) in G such that G/<(a, b)> is torsion-free.

b.) Let H_1 = <(a, 0)> and H_2 = <(0, b)>. Prove that G/(H_1 x H_2) = Z/<gcb(a, b)> x Z/<lcm(a, b)>.

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \underset{\substack{\beta C_2 + C_1 \\ \mapsto C_1}}{\overset{\substack{\alpha R_1 + R_2 \\ \mapsto R_2}}{\sim}} \begin{pmatrix} a & 0 \\ \gcd(a,b) & b \end{pmatrix} \underset{\substack{R_1 - dR_2 \\ \mapsto R_1}}{\sim} \begin{pmatrix} 0 & -db \\ \gcd(a,b) & b \end{pmatrix}$$

$$\alpha a + \beta b = \gcd(a,b)$$

$$a = d \cdot \gcd(a,b)$$

$$ab = db \cdot \boxed{\gcd(a,b)}$$

$$\| $$

$$\boxed{\gcd(a,b)} \cdot \mathrm{lcm}(a,b)$$

$$\sim \begin{pmatrix} \boxed{\gcd(a,b) \quad \boxed{0}} \\ 0 \quad \boxed{\mathrm{lcm}(a,b)} \end{pmatrix} = SNF(A)$$

$$\boxed{R_1 \leftrightarrow R_2}$$

$$\boxed{-\frac{b}{\gcd} C_1 + C_2 \mapsto C_2}$$

$$\boxed{-R_2 \mapsto R_2}$$

The column space of A is H_1 x H_2; the column space of SNF(A) is <gcd(a, b)> x <lcm(a, b)>. By the theory of Smith Normal Form, G/(H_1 x H_2) is isomorphic to Z/<gcd(a, b)> x Z/<lcm(a, b)>.

$$\mathbb{Z}_a \times \mathbb{Z}_b \cong \mathbb{Z}_{ab}^{\,*} \quad \text{if } \gcd(a,b) = 1$$

$$a = p_1^{e_1} \cdots p_k^{e_k}$$

$$b = p_1^{f_1} \cdots p_k^{f_k}$$

$$G/(H_1 \times H_2) \cong \mathbb{Z}_a \times \mathbb{Z}_b$$

$$\cong \mathbb{Z}_{p_1^{e_1}} \times \cdots \times \mathbb{Z}_{p_k^{e_k}} \times \mathbb{Z}_{p_1^{f_1}} \times \cdots \times \mathbb{Z}_{p_k^{f_k}}$$

$$\cong \prod_{i=1}^{k} \mathbb{Z}_{p_i^{\min\{e_i, f_i\}}} \times \prod_{i=1}^{k} \mathbb{Z}_{p_i^{\max\{e_i, f_i\}}}$$

$$\varphi : \mathbb{Z}\times\mathbb{Z} \longrightarrow \mathbb{Z}/\langle d\rangle \times \mathbb{Z}/\langle l\rangle$$

$$(m,n) \longmapsto (m+\langle d\rangle,\ n+\langle l\rangle)$$

$(dk,\ lt) \mapsto 0$

$m+\langle d\rangle = 0$, $n+\langle l\rangle \neq 0$

$\ker\varphi \supseteq l\mathbb{Z}$

$\mathbb{Z}\times\mathbb{Z}$

$l\mathbb{Z} \oplus l\mathbb{Z} \mapsto 0$

0

$d\mathbb{Z} \leq a\mathbb{Z}$

$a\mathbb{Z} \oplus b\mathbb{Z}$

$= l\mathbb{Z}$

$H_1 \times H_2$

$d\mathbb{Z} \leq a\mathbb{Z}$

$\leq l\mathbb{Z}$

$\cancel{l\mathbb{Z} \in}$